Since $[H]$ is an nm row by nm column nonsingular matrix, the inverse matrix $[H^{-1}]$ exists. From Eq. (33), the Lagrange multiplier $\{\lambda_{pq}\}$ can be expressed in Eq. (35) as

$$\{\lambda_{pq}\} = [H^{-1}]\Big[[D]\{k_{A_{ij}}\} - [E]\Big(\{m_{A_{ij}}\} - \{m_{0_{ij}}\}\Big)\Big] \tag{35}$$

Substituting Eq. (35) into Eqs. (21) and (22) yields the desired mass and stiffness matrices

$$\{k_{ij}\} = \{k_{A_{ij}}\} - \Big([U] + [w]\Big)[H^{-1}]\Big[[D]\{k_{A_{ij}}\} - [E]\Big(\{m_{A_{ij}}\} - \{m_{0_{ij}}\}\Big)\Big] \tag{36}$$

$$\{m_{ij}\} = \{m_{A_{ij}}\} - \{m_{0_{ij}}\} + \Big([G_1] + [G_2] - [G_3] - [G_4]\Big) \times [H^{-1}]\Big[[D]\{k_{A_{ij}}\} - [E]\Big(\{m_{A_{ij}}\} - \{m_{0_{ij}}\}\Big)\Big] \tag{37}$$

Therefore, from Eqs. (36) and (37), the corrected mass and stiffness matrices, satisfying both the orthogonality requirement and the eigenvalue equation, are obtained using the element modification method. The interaction terms between the mass and stiffness matrices can be directly determined from Eqs. (36) and (37).

Conclusions

An analytical dynamic model modification has been achieved using the element correction method. This method corrects both the mass and stiffness matrices simultaneously while enforcing the orthogonality and eigenvalue equation constraints. The interaction effects between the mass and stiffness matrices that are given in the final equations can be very important to the analytical dynamic engineer. Also, no iteration is required in order to ensure that the desired mass and stiffness matrices satisfy the dynamic constraint. This method is very useful to improve the analytical model based on an incomplete set of modal test data.

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Application of Load-Dependent Vectors Bases for Dynamic Substructure Analysis

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I. Introduction

THE classical approach to dynamic substructuring has been used to describe the internal motions of each substructure by a linear combination of substructure modes, with the implicit assumption that these modes satisfy a certain substructure eigenvalue problem. Three basic variants of the method have been developed depending on whether the modes of each substructure are obtained with its interface held fixed, free, or loaded. Since it is not possible to define a unique eigenvalue problem for a given substructure and because none of these methods can yield exact results for the actual structure using a truncated set of exact mode shapes, the advisability of using exact substructure mode shapes can be seriously questioned. The suggestions for improvements should therefore be directed toward two basic questions: how to select a set of substructure modes, and how to enforce geometric compatibility at substructure boundaries.

A new method of dynamic analysis for structural systems subjected to fixed spatial distribution of the dynamic load was recently introduced by Wilson et al.[1] as an economic alternative to the classical mode-superposition technique. The method of solution is based on a transformation to a reduced system of generalized Ritz coordinates using load-dependent transformation vectors. By using the superposition of load-dependent vectors, static correction components similar to those of the classical mode-acceleration method are directly computed. New computational variants used to generate load-dependent vectors have been presented recently.[2] The method has also been applied in the context of dynamic substructuring as a direct extension of the fixed interface synthesis of component modes, replacing substructure eigenvectors by load-dependent Ritz transformation vectors or Lanczos vectors.[3]

The purpose of this Note is to discuss practical implementation aspects of this dynamic substructuring method related to the algorithm used to generate transformation vectors and the corresponding evaluation of error norms. A new computational variant that combines the subspace generation of load-dependent vectors and the static condensation technique is developed to compute global modes which are not a summation of local modes or a combination of substructure modes.

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Finally, a numerical example of a simple cantilever structure is presented to illustrate the relative performance of the proposed solution methods.

Generation of Load-Dependent Vectors

The proposed algorithm used to generate load-dependent vectors for dynamic substructuring is shown in Table 1. It is a subspace iteration variant of the original algorithm presented by Wilson et al.[1] The spatial load distributions assigned to each substructure should be used to obtain the corresponding load-dependent vectors. To model multispatial transient loads, the selection of the initial loading distributions should be made with the help of a linear independency criterion to reduce the repetitive load patterns to a minimum and to favor the generation of an orthogonal basis. To obtain an accurate structural response in all substructures, load-dependent vectors should also be included for substructures that are not externally loaded. If the dynamic model has first been validated by a static analysis, then an appropriate choice will be to initiate the algorithm for the unloaded substructures from pseudoinertial loads obtained from

$$[F] = [M][U]_o^s \tag{1}$$

where $[M]$ is a specified substructure mass matrix and $[U]_o^s$ is the $[M]$-orthonormalized static deflected shapes obtained from a static substructure analysis of the system subjected to the spatial distributions of the dynamic load. A similar procedure can be used to determine the free-free modes of the combined structure, for which no external loads exist in any substructure by assuming a uniform displacement field for $[U]_o^s$.

One of the important aspects in the application of coordinate reduction techniques using transformation vectors pertains to the number of vectors that must be retained in the analysis. In the case of arbitrary dynamic loads, a measure of the loading representation based on the Euclidean norm can be used to monitor the relative amount of the specified dynamic loads that will be included in the solution:

$$e_{j,r} = \left| \frac{1 - |\{f_j(s)\} - \{f_{j,r}(s)\}|_2}{|\{f_j(s)\}|_2} \right| *100 \tag{2}$$

where $\{f_j(s)\}$ is the jth spatial distribution pattern of the specified dynamic loads, and

$$\{f_{j,r}(s)\} = [M][X][X]^T\{f_j(s)\} \tag{3}$$

is the representation of $\{f_j(s)\}$ obtained by the truncated $[M]$-orthonormal vector basis, $[X]$. It is also possible to define an error estimate that can be used to indicate when the spectral content of the starting vectors will be exhausted. At any step of the computations, the Euclidean norm of the static residual, written as $|[M]\{U\}_i|_2$, can be compared to the norm or an average of the norms of the initial static solutions, $|[M]\{U\}_o|_2$, for that purpose. When the ratio

$$(|[M]\{U\}_i|_2)/(|[M]\{U\}_o|_2) \tag{4}$$

will drop below the order of the numerical roundoff value of the computer, the algorithm will potentially become able to generate null vectors. If pseudo-inertial loads are used, these error norms will provide an indication of the effective mass represented by the vector basis.

Spatial error norms can be computed at the substructure level to ensure an adequate representation of the specified substructure loading. The error in the final structural response of the complete system will in general be small if there is a good representation of the loading in each substructure. However, since there is no interaction between the substructures when the transformation vectors are being generated, there is still an uncertainty related to the synthesis of errors that cannot be easily quantified.

Table 1 Subspace generation of load-dependent vectors

A. Dynamic Equilibrium Equations
$[M]\{\ddot{U}\} + [C]\{\dot{U}\} + [K][U] = [F(s)]\{g(t)\}$

B. Initial Calculations
1. Triangularize stiffness matrix
   $[K] = [L][D][L]^T$
2. Solve for static response
   $[K][U]_o = [F(s)]$
3. M-Orthonormalize $[U]_o$

C. Generate Ritz Vectors $[X]_i$, $i = 1, \ldots, r-1$
1. Solve for $[X]_i$
   $[K][X]_i = [M][U]_{i-1}$
2. Solve reduced subspace eigenproblem
   $[\bar{K}][Z] = [\bar{M}][Z][V]$
   where $[\bar{K}] = [X]_i^T[K][X]_i$ and $[\bar{M}] = [X]_i^T[M][X]_i$
3. M-Orthogonalize $[X]_i$ against previous blocks (Gram-Schmidt)
4. M-Orthonormalize block $[X]_i$
5. Remove new Ritz vectors from static block
   $[U]_i = [U]_{i-1} - [X]_i([X]_i^T[M][U]_{i-1})$
6. Evaluate error, stop, or repeat

D. Add Static Block Residual $[U]_{r-1}$ as Static Correction Terms $[X]_r$

E. Make Vectors Stiffness Orthogonal (Optional)
1. Solve $r \times r$ eigenvalue problem
   $([K]^* - \bar{w}^2[I])[Y] = [O]$
   where $[K]^* = [X]^T[K][X]$
2. Calculate orthogonal Ritz vectors $[\phi] = [X][Y]$

Dynamic Ritz Condensation Algorithm

The load-dependent vectors are generated from a recurrence relationship that uses a static solution to fictitious inertial loadings. It is therefore appropriate to think about the generation of load-dependent vectors using a static substructuring method. The generation of global load-dependent vectors using static substructuring will thus be able to take into account the effect of adjoining substructures exactly. Moreover, with this approach the spatial error norms can be computed on the complete systems without any synthesis approximations. This method will be introduced as the dynamic Ritz condensation (DRC) algorithm which can be summarized as follows:

1) Generate partitioned substructures stiffness, mass, and load matrices.

2) Assemble and store complete system mass matrix (optional).

3) Reduce substructure stiffnesses and assemble global stiffness matrix.

4) Solve for initial static vector by static substructuring.

5) Compute load-dependent basis, from the solution of new vectors using static substructuring at step C.1 of the algorithm in Table 1.

Numerical Applications

To evaluate the proposed dynamic substructuring algorithm, a 40-DOF shear beam cantilever structure was divided into four substructures, SSA, SSB, SSC, and SSD, as shown in Fig. 1. The structural response to the concentrated load applied to SSD was first calculated by the method of component mode synthesis adding generalized coordinates only in SSD. Exact eigenvectors combined with residual attachment modes, and load-dependent vectors were used in the analyses.

The structural response in terms of the maximum relative error in SSD beam shear forces is shown in Fig. 2. A fully converged solution was defined when the maximum error in

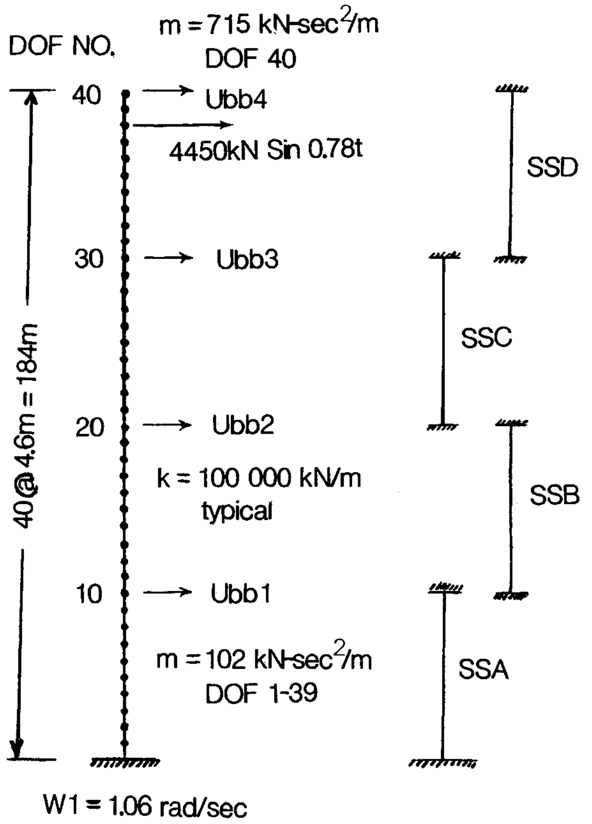

Fig. 1 Forty DOF shear beam structure.

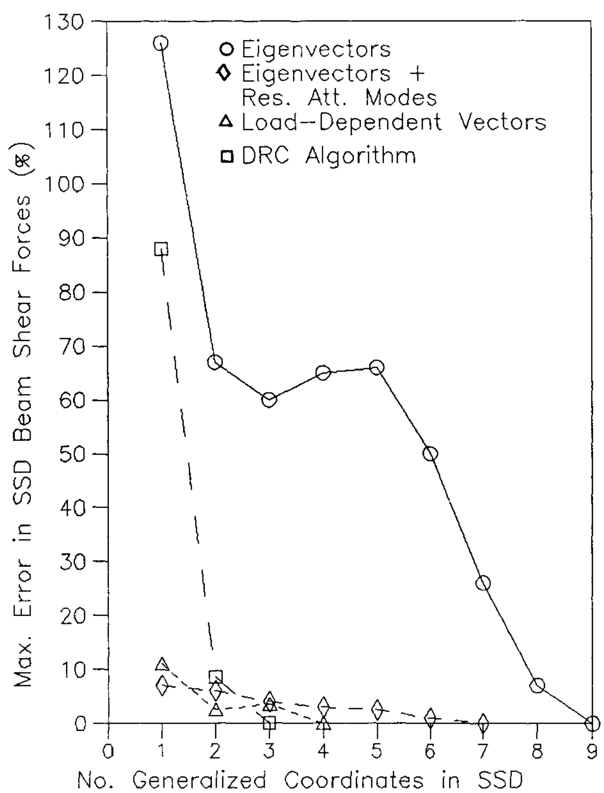

Fig. 2 Convergence characteristics of various dynamic substructuring methods.

beam shear force was found to be less than 1% of the exact mathematical solution. It was observed that the response in SSA, SSB, and SSC can be considered almost independent of the number of generalized coordinates retained to represent the internal behavior of SSD. A constant maximum error of 9% was obtained from any vector basis.

It was found that the addition of 1 generalized coordinate in SSA, SSB, and SSC was sufficient to obtain convergence from any basis. In summary, the total number of degrees of freedom (boundary and internal) that were uncoupled in the eigensolution of the reduced global matrix system to obtain steady-state shear force convergence in all substructures was 16 for exact eigenvectors, 12 for exact eigenvectors combined with residual attachment modes, and 11 for load-dependent vectors.

The results obtained from the DRC algorithm are also reported in Fig. 2 for comparison purposes. Only 3 load-dependent vectors were necessary by this approach such that the size of the reduced eigenproblem was of order 3. The eigenproblem generated by the load-dependent synthesis method was thus larger than the eigenproblem obtained from the DRC algorithm; moreover, there are mass coupling terms in the synthesis method such that a numerical algorithm adapted to the solution of the generalized eigenproblem must be used. On the other hand, the eigenproblem obtained from the DRC algorithm is already cast in the standard form since the reduced mass matrix will correspond to the identity matrix from orthonormality conditions. For this example, the solution using the DRC algorithm was thus found to be more efficient than the load-dependent systhesis method. It is anticipated that if accurate results are required in all substructures, the DRC algorithm will be able to maintain a numerical advantage over the load-dependent synthesis method for a wide range of applications by generating reduced Ritz systems that require fewer generalized degrees of freedom for convergence.

Conclusions

Derivation of load-dependent Ritz coordinates is much less expensive than the solution of the corresponding eigenproblem. Two different methods to apply load-dependent vectors for linear dynamic substructure analysis have been investigated. In the first technique, the internal behavior of a substructure is represented by a small number of load-dependent Ritz coordinates in a component mode synthesis formulation where constraint modes are used to interface the components. For that purpose, a new subspace algorithm is recommended to generate the load-dependent vectors. In a second technique, called the DRC algorithm, it was shown that it is possible to generate efficiently the load-dependent Ritz basis of the complete structure from the structural properties of each substructure by using an iterative version of the familiar static condensation algorithm. Numerical applications on a simple system were used to illustrate that these techniques exhibit a better rate of convergence than the methods using eigenvectors as transformation vectors. The DRC algorithm is recommended for the analysis of large systems since it produces reduced systems of equations that can be solved more efficiently than those obtained by the synthesis of load-dependent component vectors.

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